

Single-parameter Mechanism design: Sponsored Search Auctions

G. Amanatidis

Based on slides by A. Voudouris

Single-item auctions

- A seller with **one item for sale**
- n **agents**
- Each agent i has a **private value** v_i for the item
 - This value represents the **willingness-to-pay** of the agent; that is, v_i is the maximum amount of money that agent i is willing to pay in order to buy the item
- The utility of each agent is **quasilinear in money**:
 - If agent i loses the item, then her utility is 0
 - If agent i wins the item at price p , then her utility is $v_i - p$

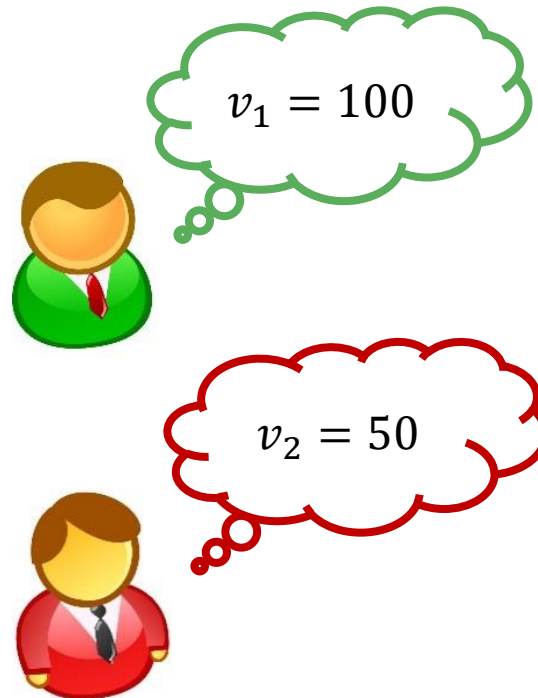


Single-item auctions

- General structure of an auction:
 - **Input:** every agent i submits a bid b_i (*agents = bidders*)
 - **Allocation rule:** decide the winner
 - **Payment rule:** decide a selling price
- Deciding the winner is easy: the highest bidder
- Deciding the selling price is more complicated
 - A selling price of 0, creates a competition among the bidders as to who can think of the highest number
- We are interested in payment rules that incentivize the bidders to bid their true values
 - **Truthful auctions that maximize the social welfare**

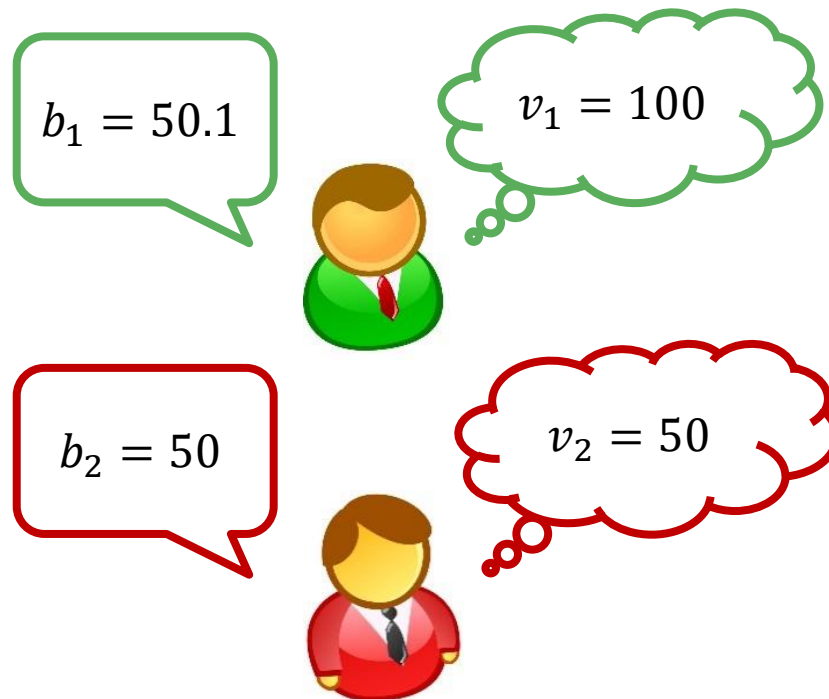
First-price auction

- **Allocation rule:** the winner is the highest bidder
- **Payment rule:** the winner pays her bid
- Is this a truthful auction?



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Second-price auction

- **Allocation rule:** the winner is the highest bidder
- **Payment rule:** the winner pays the second highest bid

Theorem [Vickrey, 1961]

In a second-price auction

- (a) it is a dominant strategy for every bidder i to bid $b_i = v_i$, and
 - (b) every truthtelling bidder gets non-negative utility
- (b) is obvious:
 - the selling price is at most the winner's bid, and the bid of a truthtelling bidder is equal to her true value

Second-price auction

- For (a), our goal is to show that the utility of bidder i is maximized by bidding v_i , no matter what v_i and the bids of the other bidders are
- Second highest bid: $B = \max_{j \neq i} b_j$
- The utility of bidder i is either 0 if $b_i < B$, or $v_i - B$ otherwise

Case I: $v_i < B$

- Maximum possible utility = 0
- Achieved by setting $b_i = v_i$

Case II: $v_i \geq B$

- Maximum possible utility = $v_i - B$
- Bidder i wins the item by setting $b_i = v_i$



Sponsored search auctions

Google buy a computer

All Shopping Maps Images News More Settings Tools

About 4,510,000,000 results (0.72 seconds)

1st 0.18\$ 22¢

PCSpecialist | Buy your New Computer | PCSpecialist.co.uk
Ad www.pcspecialist.co.uk/
★★★★★ Rating for pcspecialist.co.uk: 4.8 - 774 reviews
Configure your new custom computer to your exact requirements. Next day PCs also available!

All-In-One Computers Choose from Our Range of Intel - Based AIO PC Systems.
Game-Based Computers View Our Recommendations for PCs Based on Your Favourite Games.

2nd 0.145\$ 18¢

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3rd 0.18 9¢

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4th 0.5 3¢

Desktop PCs at PC World | Free Delivery On All Orders | PCWorld.co.uk
Ad www.pcworld.co.uk/Desktops
Collect In-Store Available. Fast & Secure Checkout. Intel® Core™ Processors Inside.

Handwritten notes: CTR, $v_{dell} = 0.5$

CTR

0.36

0.29

0.18

0.5

Sponsored search auctions

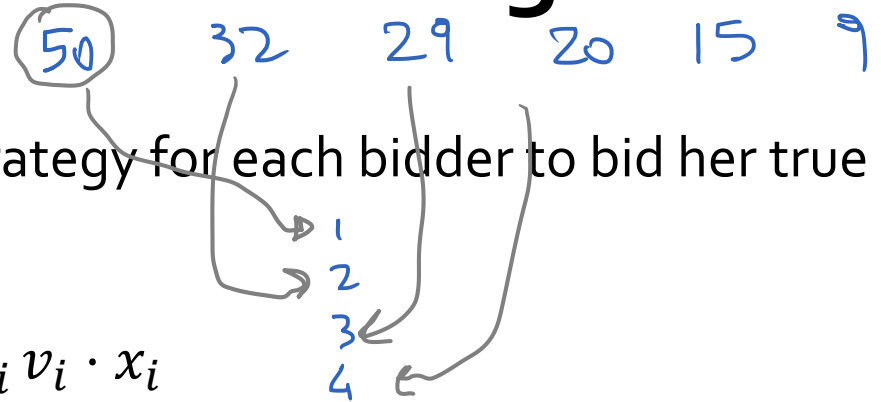
The screenshot shows a Google search for "buy a computer". The search bar contains the text "buy a computer" and a microphone icon. Below the search bar are navigation links: All, Shopping, Maps, Images, News, More, Settings, and Tools. The search results show "About 4,510,000,000 results (0.72 seconds)". Below the search results, there are three sponsored search auctions. The first is for "Dell PC Servers | Powered By Intel Xeon | dell.com" with a phone number "0333 258 0993". The second is for "Buy a PC with Cyberpower UK | Finance Options" with a phone number "0333 323 7776". The third is for "Desktop PCs at PC World | Free Delivery On All Orders | PCWorld.co.uk" with a phone number "0333 323 7776".

- In 2011 Google's revenue was almost 40.000.000.000 usd
- 96% of this was generated by sponsored search auctions

Sponsored search auctions

- k advertising **slots**
- n **bidders** (advertisers) who aim to occupy a slot
- Slot j has a **click-through-rate** (CTR) a_j
 - The CTR of a slot represents the **probability that the ad placed at this slot will be clicked on**
 - Assumption: the CTRs are independent of the ads that occupy the slots
- The slots are ranked so that $a_1 \geq \dots \geq a_k$
- Each bidder i has a **private value** v_i **per click**
 - Bidder i derives utility $a_j \cdot v_i$ from slot j

Sponsored search auctions: goals



- **Truthfulness**: It is a dominant strategy for each bidder to bid her true value
- **Social welfare maximization**: $\sum_i v_i \cdot x_i$
 - x_i is the CTR of the slot that bidder i is assigned to, or 0 otherwise
- **Poly-time execution**: running the auction should be quick

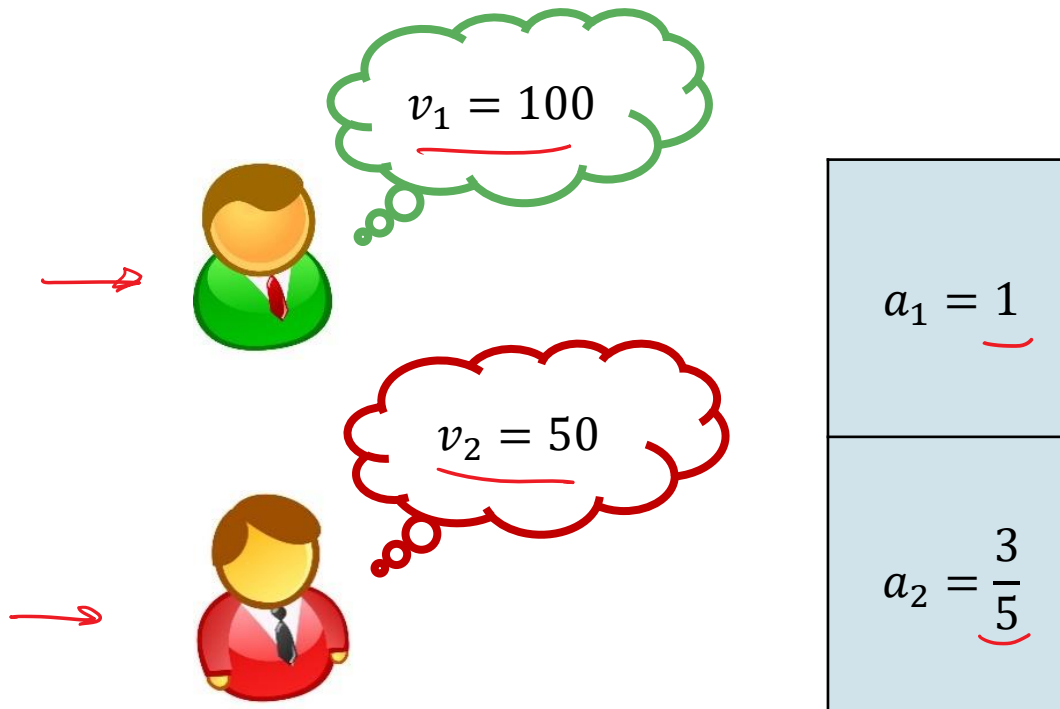
$\theta(n)$

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- **Poly-time execution:** running the auction should be quick
- If the bidders are truthful, then maximizing the social welfare is easy: sort the bidders in decreasing order of their bids
- So, the problem is to incentivize them to be truthful, again
- Can we extend the ideas we exploited for single-item auctions?

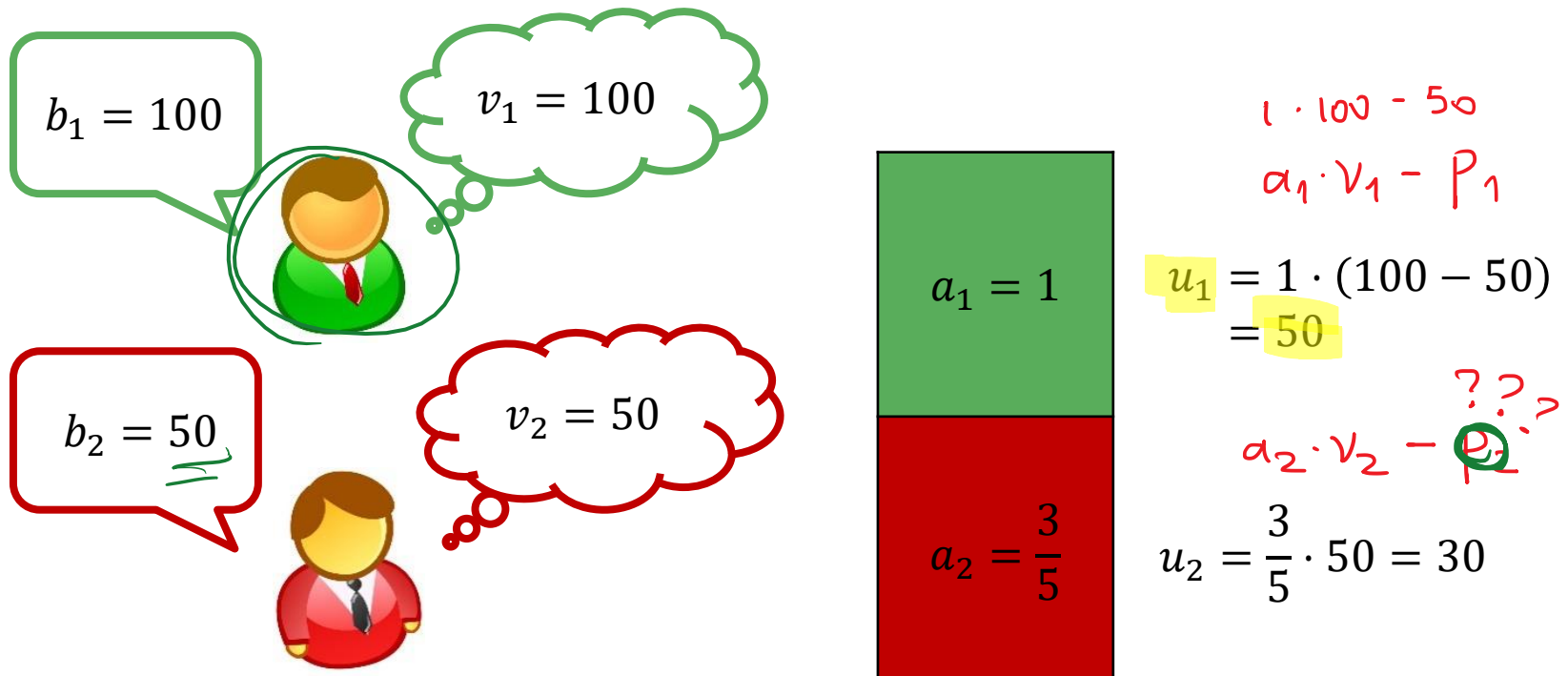
Generalized second-price auction

- **Allocation rule:** sort the bidders in decreasing order of their bids and rename them so that $b_1 \geq \dots \geq b_n$
- **Payment rule:** every bidder $i \leq k$ (who is assigned at slot i) pays the next highest bid b_{i+1} per click, and every bidder $i > k$ pays 0



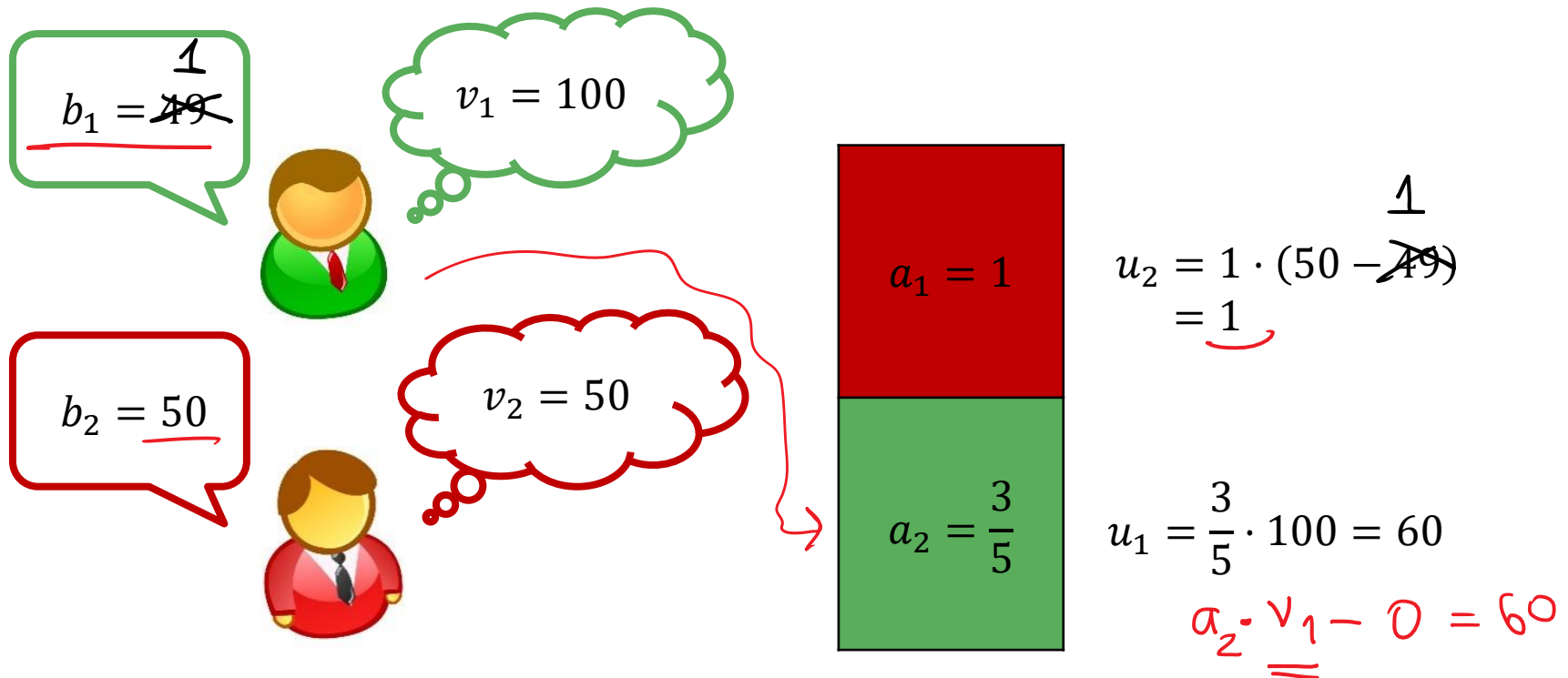
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Myerson's Lemma

- That didn't work for sponsored search auctions, so what now?
- Let's try to see how the optimal truthful auction should look like, for any single parameter environment
- Input by bidders: $\mathbf{b} = (b_1, \dots, b_n)$
- Allocation rule: $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), \dots, x_n(\mathbf{b}))$
- Payment rule: $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), \dots, p_n(\mathbf{b}))$
- The utility of bidder i is $u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$
- Focus on payment rules such that $p_i(\mathbf{b}) \in [0, b_i \cdot x_i(\mathbf{b})]$
 - $p_i(\mathbf{b}) \geq 0$ ensures that the seller does not pay the bidders
 - $p_i(\mathbf{b}) \leq b_i \cdot x_i(\mathbf{b})$ ensures non-negative utility for truthful bidders

$$x_5(\mathbf{b}) = \begin{cases} \rightarrow 3^{\text{rd}} \text{ slot} \\ \rightarrow 5^{\text{th}} \text{ slot} \\ \rightarrow \text{no slot} \end{cases}$$

Myerson's Lemma

- An allocation rule \mathbf{x} is **implementable** if there exists a payment rule \mathbf{p} such that (\mathbf{x}, \mathbf{p}) is a truthful auction
- An allocation rule \mathbf{x} is **monotone** if for every bidder i and bid vector \mathbf{b}_{-i} , the allocation $x_i(z, \mathbf{b}_{-i})$ is non-decreasing in the bid z of bidder i

Lemma [Myerson, 1981]

→ (a) An allocation rule \mathbf{x} is implementable if and only if it is monotone

→ (b) For every ~~allocation~~ ^{monotone} rule \mathbf{x} , there exists a unique payment rule \mathbf{p} such that (\mathbf{x}, \mathbf{p}) is a truthful auction

can give a truthful mech.

Proof of Myerson's Lemma

- Fix a bidder i , and the bids \mathbf{b}_{-i} of the other bidders
- Given that these quantities are now fixed, we simplify our notation:
 - $x(z) = x_i(z, \mathbf{b}_{-i})$
 - $p(z) = p_i(z, \mathbf{b}_{-i})$
 - $u(z) = u_i(z, \mathbf{b}_{-i})$
- The idea:
 - assuming (\mathbf{x}, \mathbf{p}) is a truthful auction, the bidder has no incentive to unilaterally deviate to any other bid
 - This will give us a relation between \mathbf{x} and \mathbf{p} , which we can use to derive an explicit formula for \mathbf{p} as a function of \mathbf{x}

Proof of Myerson's Lemma

- Consider two bids $0 \leq z < y$ and assume x is implementable by p

- True value = z , deviating bid = y :

$$\begin{aligned} u(z) \geq u(y) &\Leftrightarrow z \cdot x(z) - p(z) \geq z \cdot x(y) - p(y) \\ &\Leftrightarrow p(y) - p(z) \geq z \cdot (x(y) - x(z)) \end{aligned}$$

- True value = y , deviating bid = z :

$$\begin{aligned} u(y) \geq u(z) &\Leftrightarrow y \cdot x(y) - p(y) \geq y \cdot x(z) - p(z) \\ &\Leftrightarrow p(y) - p(z) \leq y \cdot (x(y) - x(z)) \end{aligned}$$

Proof of Myerson's Lemma

- Combining these two, we get:

$$z \cdot (x(y) - x(z)) \leq p(y) - p(z) \leq y \cdot (x(y) - x(z))$$

- This also implies that

$$(y - z) \cdot (x(y) - x(z)) \geq 0$$

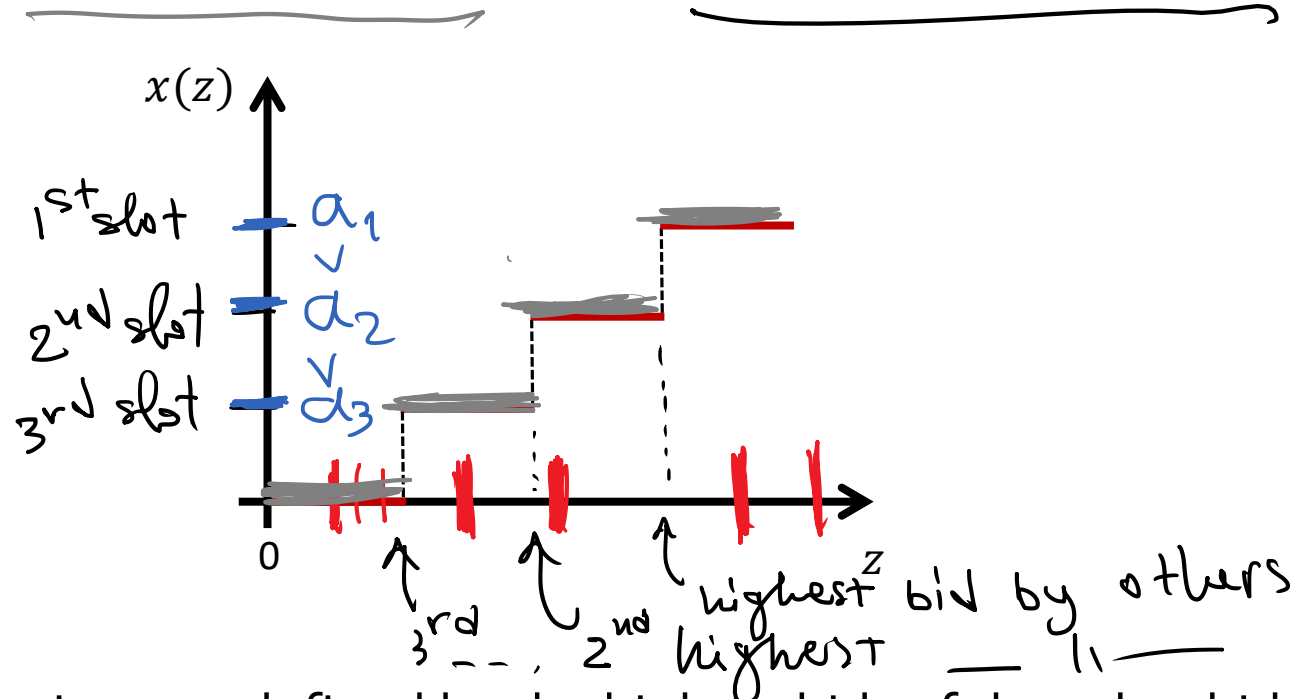
- Since $0 \leq z < y$, this is possible if and only if x is monotone so that $y - z > 0$ and $x(y) - x(z) > 0$

\Rightarrow (a) is now proved

Proof of Myerson's Lemma

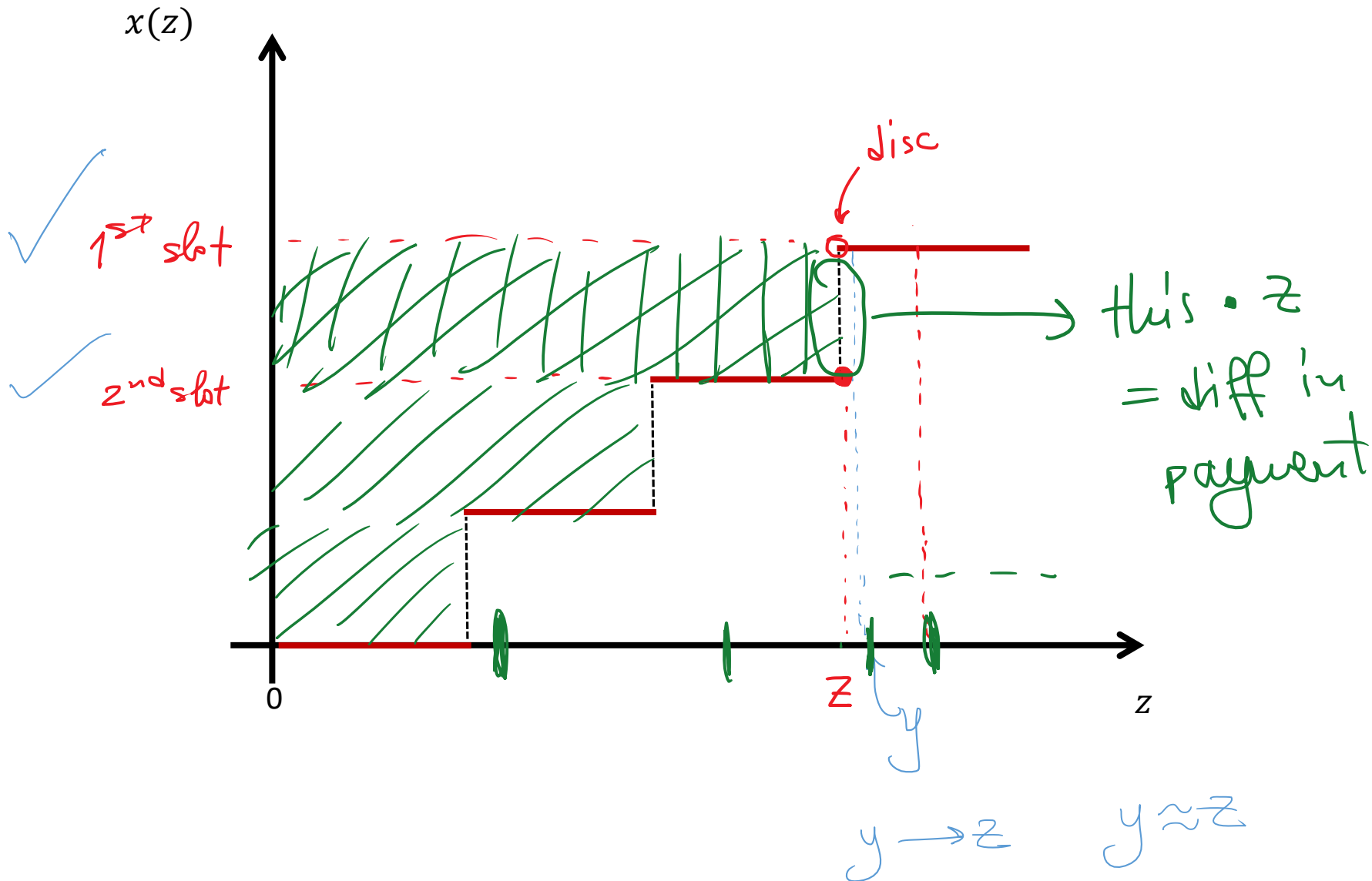
3 slots

- We can now assume that x is monotone
- Assume x is piecewise constant, like in sponsored search auctions



- The break points are defined by the highest bids of the other bidders

Proof of Myerson's Lemma



Proof of Myerson's Lemma

If z is at a "jump":

$$z \cdot \underbrace{(x(y) - x(z))}_{\text{red underline}} \leq p(y) - p(z) \leq \overset{z}{\underset{\uparrow}{y}} \cdot \underbrace{(x(y) - x(z))}_{\text{red underline}}$$

- By fixing z and taking the limit as y tends to z , we have that

jump of p at $z = z \cdot (\text{jump of } x \text{ at } z)$

$$p(z+\varepsilon) - p(z) = z \left(x(z+\varepsilon) - x(z) \right)$$

Proof of Myerson's Lemma

$$z \cdot (x(y) - x(z)) \leq p(y) - p(z) \leq y \cdot (x(y) - x(z))$$

- By fixing z and taking the limit as y tends to z , we have that

$$\text{jump of } p \text{ at } z = z \cdot (\text{jump of } x \text{ at } z)$$

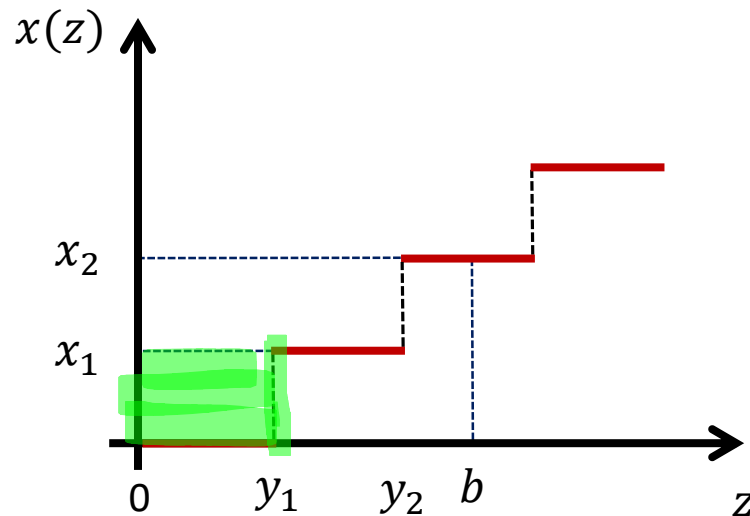
- Therefore, we can define the payment of the bidder as

$$p(b) = \sum_{y \in [0, b]} y \cdot (\text{jump of } x \text{ at } y)$$

where y enumerates all break points of x in $[0, b]$

Proof of Myerson's Lemma

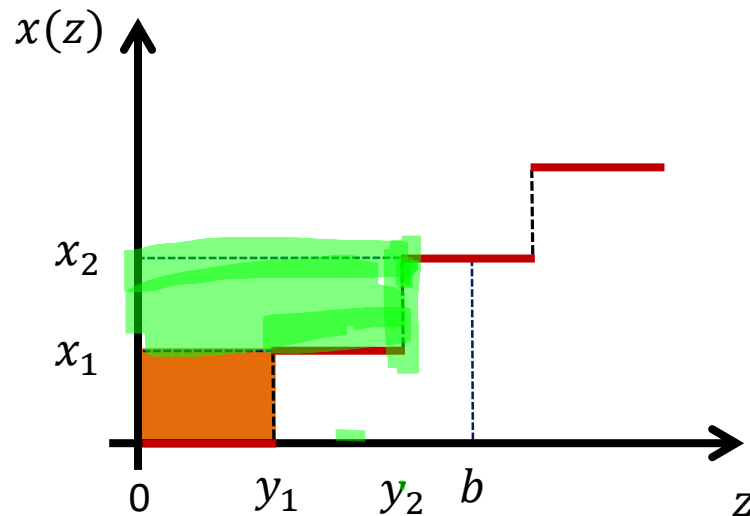
- Example:



$$p(b) = \sum_{y \in [0, b]} y \cdot (\text{jump of } x \text{ at } y)$$

Proof of Myerson's Lemma

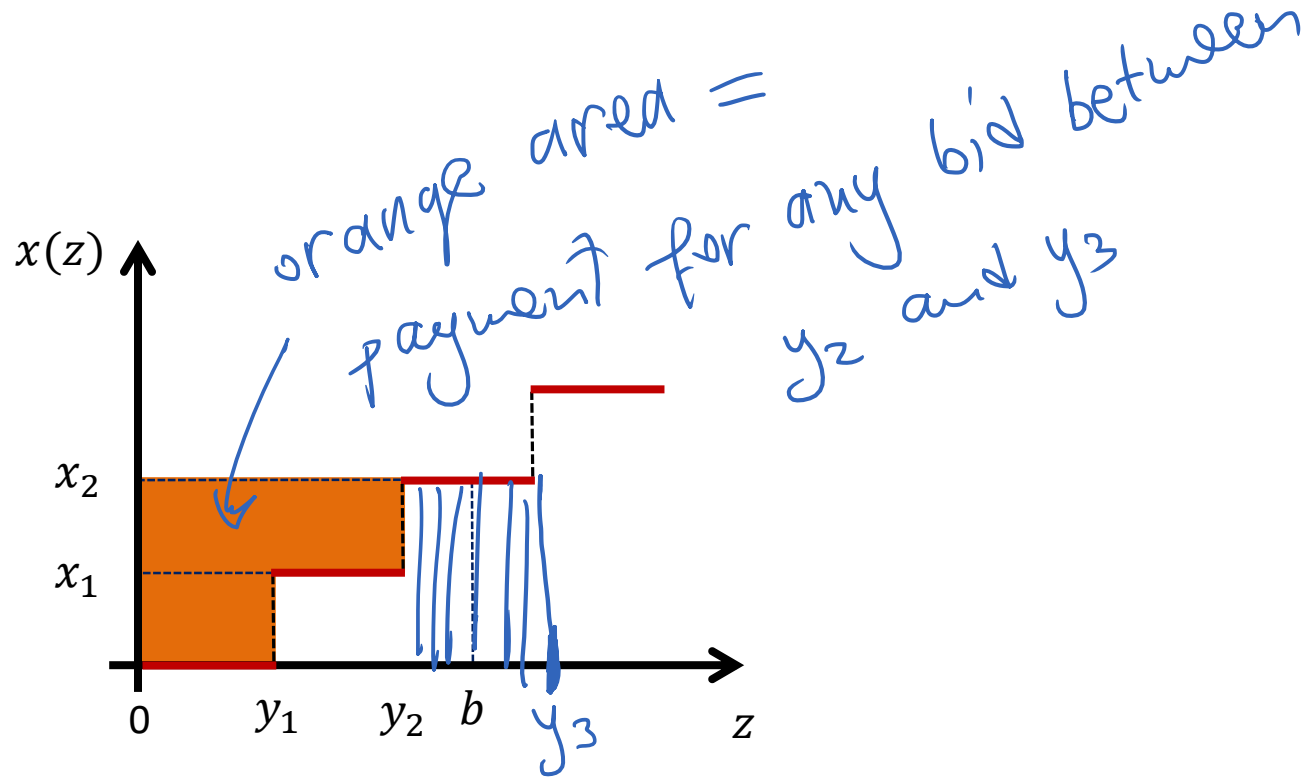
- Example:



$$p(b) = \sum_{y \in [0, b]} y \cdot (\text{jump of } x \text{ at } y) = y_1 \cdot x_1$$

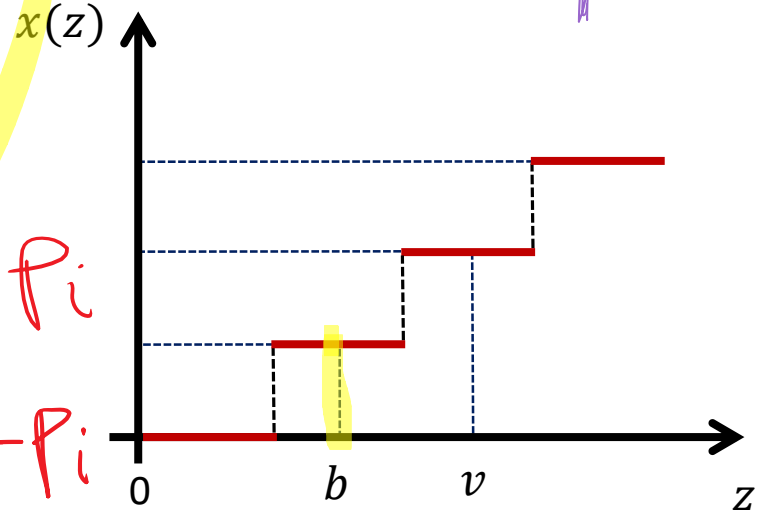
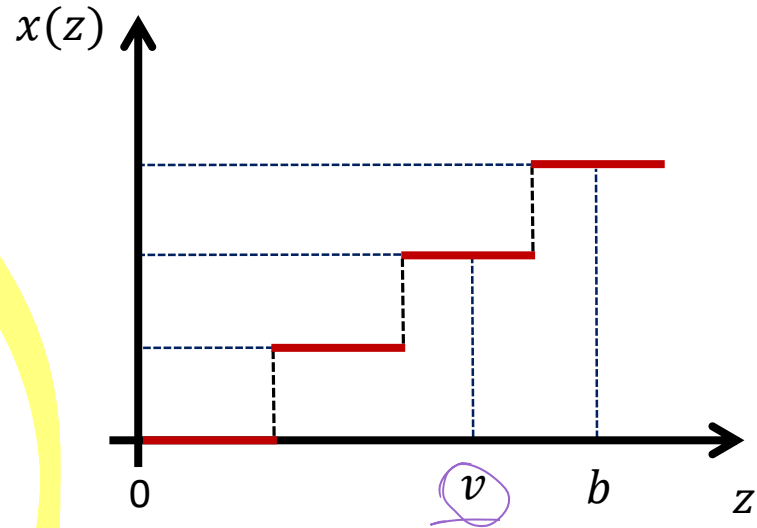
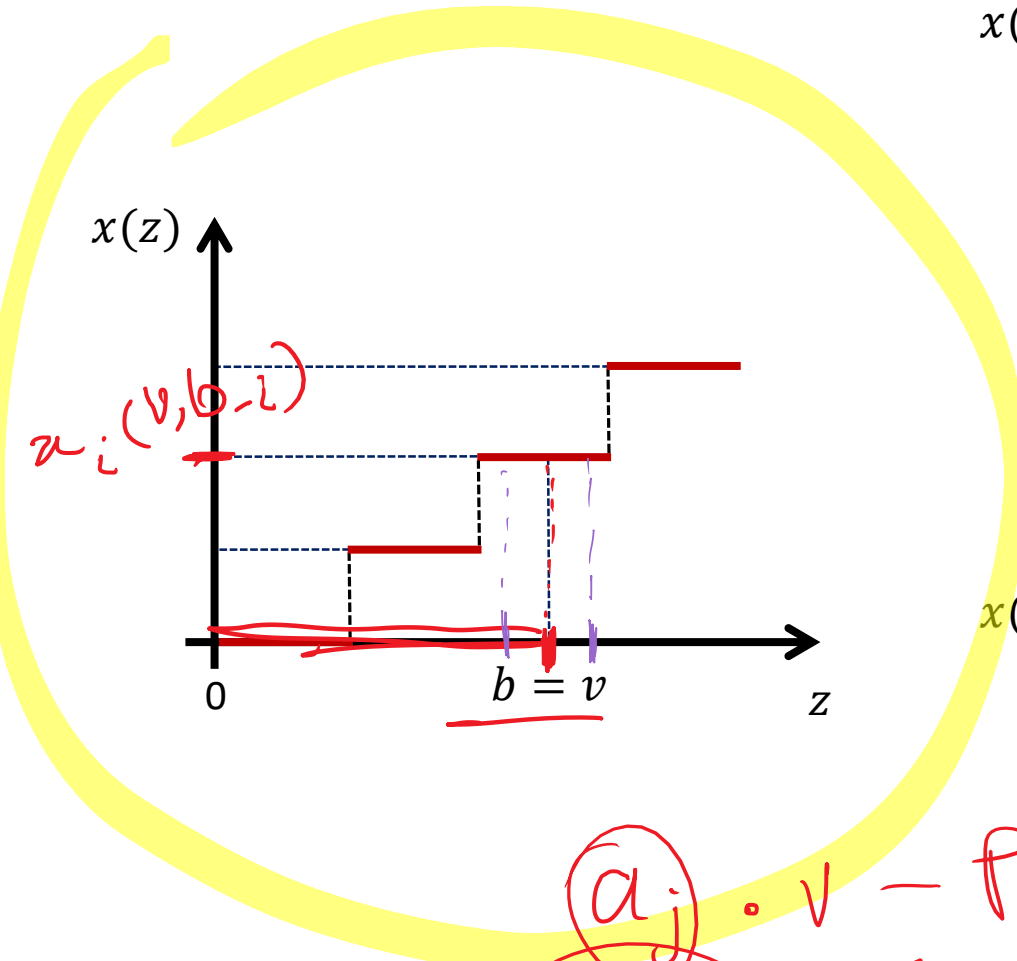
Proof of Myerson's Lemma

- Example:



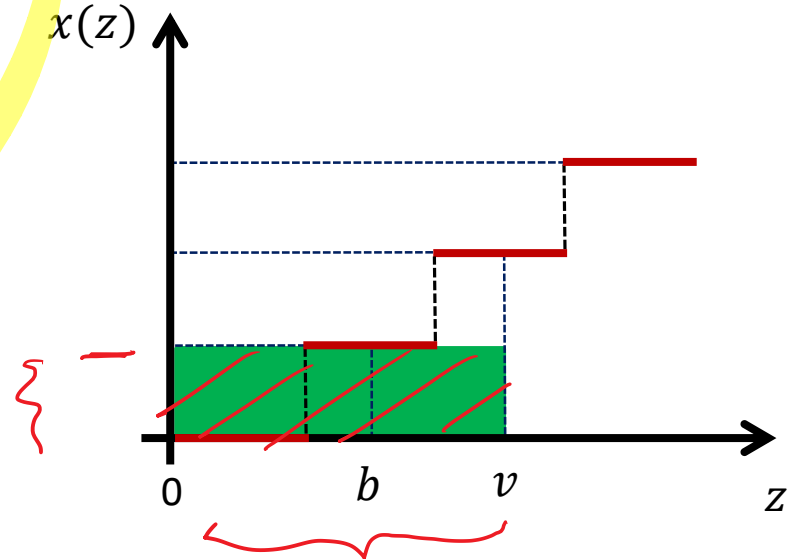
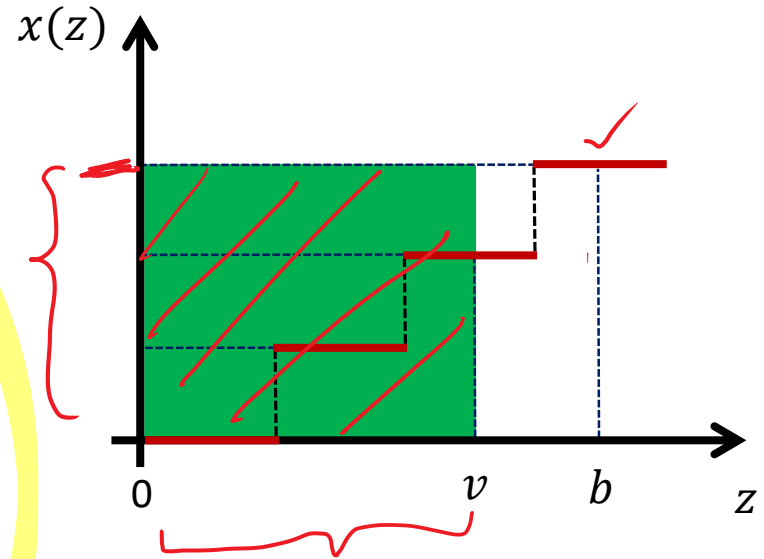
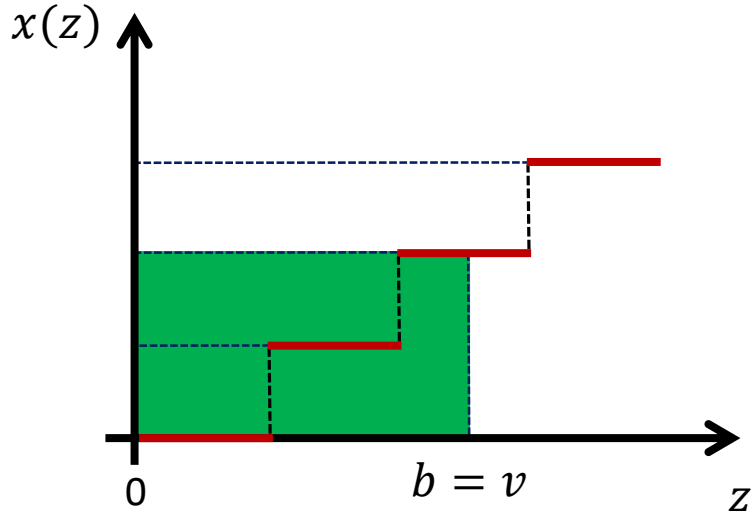
$$p(b) = \sum_{y \in [0, b]} y \cdot (\text{jump of } x \text{ at } y) = y_1 \cdot x_1 + y_2 \cdot (x_2 - x_1)$$

Proof of Myerson's Lemma

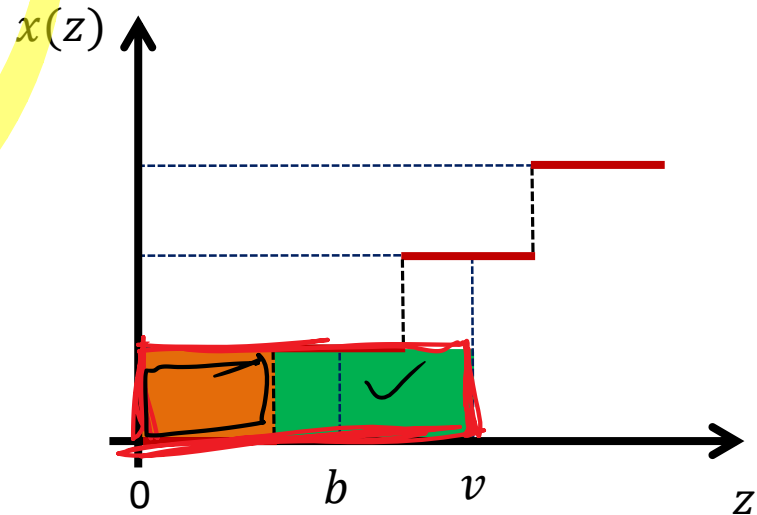
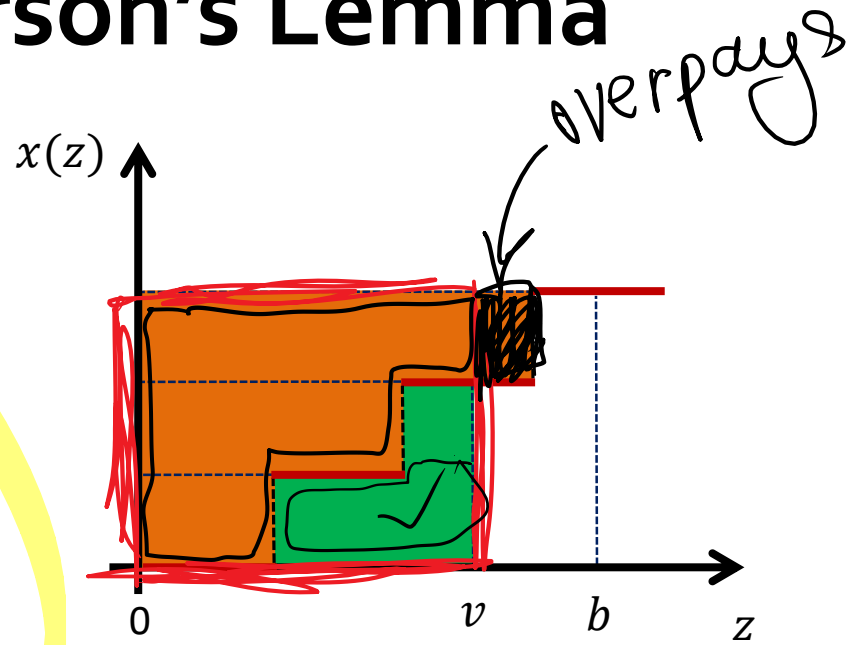
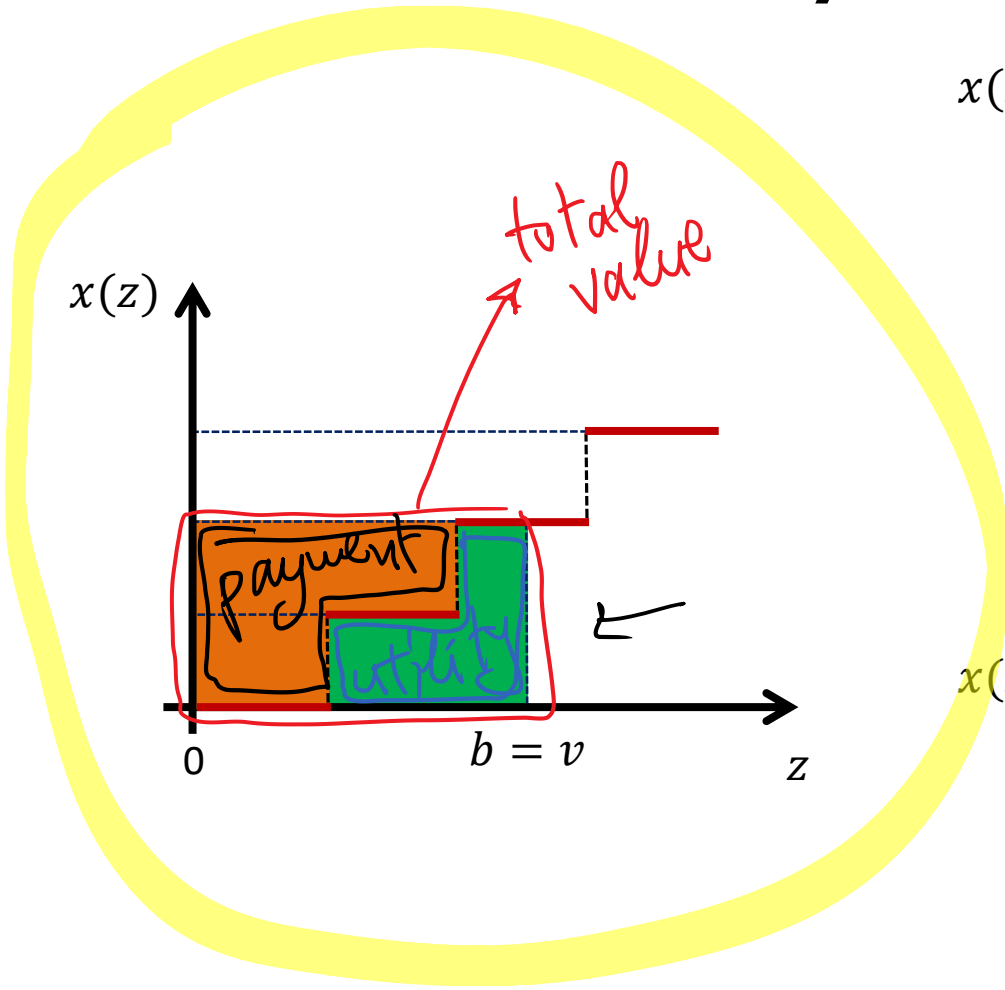


$$u_i(v, b_i) = \underbrace{x_i(v, b_i) \cdot v}_{(a_i) \cdot v} - p_i$$

Proof of Myerson's Lemma



Proof of Myerson's Lemma



Sponsored search auctions

$$p(b) = \sum_{y \in [0, b]} y \cdot (\text{jump of } x \text{ at } y)$$

- y enumerates the break points: the bids that are smaller than b
 - In other words, y enumerates the slots from worst to best
- jump of x at y : the difference in CTR between two consecutive slots
- The total payment of the i -th highest bidder is:

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=i}^k b_{j+1} (a_j - a_{j+1})$$

sum of
orange
#reds

Summary

- **Auctions:** allocation rule + payment rule
- An allocation rule is **implementable** if there exists a payment rule, so that together they define a truthful auction
- An allocation rule is **monotone**, if larger bids give more stuff
- **Single-item auctions:** first-price is not truthful, second-price is truthful and maximizes the social welfare (sells to the bidder with the highest value)
- **Sponsored search auctions:** generalized second-price auction is not truthful
- **Myerson's Lemma:** a characterization of truthful mechanisms in single-parameter environments
- Using Myerson's Lemma we can design a truthful sponsored search auction